positions in the series of natural numbers beginning with n as they do in the arithmetical progression. This is to say that if the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th, $\cdots$ terms of the progression are divisible by m_1, so also will be the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th $\cdots$ terms of the series $n, n + 1, n + 2, \cdots$, etc. Show that n may be determined as the solution of a congruence $An + B \equiv 0 \pmod{C}$ whose coefficients, A, B, are constants independent of the number and value of the m's.

Solution by Elijah Swift, University of Vermont.

By the conditions of the problem the following equations hold: (1) $a + (k - 1)d = m_1\alpha$, (2) $n + k - 1 = m_1\beta$, where α and β are integers. Eliminating k, we obtain,

$$dn - a = m_1(\beta d - \alpha). \tag{3}$$

Consequently we see that n must be a solution of the congruence $dn - a \equiv 0$ (mod L.C.M. of $m_1, m_2, \cdots$). Conversely any solution of this congruence satisfies the given condition. For let n be such a solution. Then (3) is true. We are to show that if (1) holds, (2) will also: and conversely. But adding the equations (1) and (3) we see that $d(n + k - 1)$ is divisible by m_1, and since d is prime to m_1, $n + k - 1$ is divisible by m_1. Similarly (2) implies (1).

Also solved by C. F. Gummer and Horace Olson.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

DISCUSSION.

Relating to a Construction for the Graph of a Cubic.

By Paul Capron, U. S. Naval Academy.

If a cubic is given by the equation

$$\alpha y = x^3 + \beta x^2 + \gamma x + \delta, \qquad (\alpha > 0)$$

and the origin is shifted to (x_0, y_0), where

$$x_0 = -\frac{\beta}{3}, \qquad y_0 = \frac{1}{27\alpha}(2\beta^3 - 9\beta\gamma + 27\delta),$$

the new equation is

$$\alpha y = x^3 + (\gamma - \beta^2/3)x.$$

It is convenient to distinguish three cases, according to the value of $(\gamma - \beta^2/3)$.

Case (1). $\gamma - \beta^2/3 < 0$. Write $\alpha = a^2$, $\gamma - \beta^2/3 = -c^2$; then the equation is

$$a^2 y = x^3 - c^2 x.$$

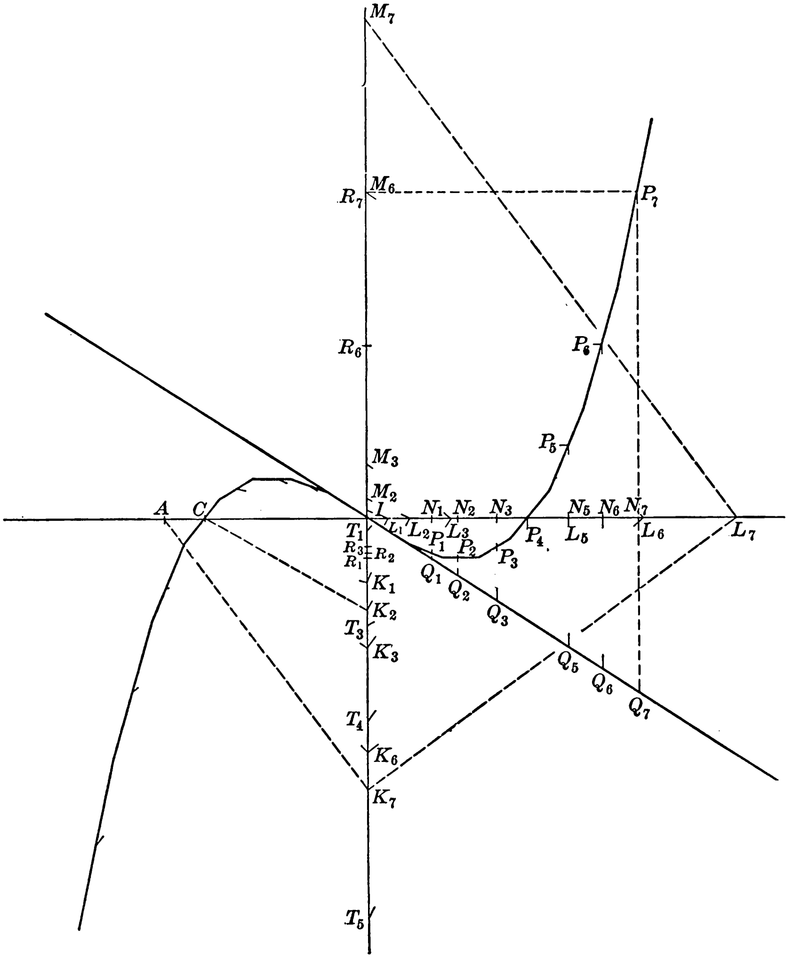
$dy/dx = 1/a^2(3x^2 - c^2)$. At the inflection, I, $(0, 0)$, we have $dy/dx = -c^2/a^2 = 1/3\alpha(3\gamma - \beta^2)$, and $d\dot{y}/dx = 0$ at $[\pm c/\sqrt{3}, \mp (2/3\sqrt{3})(c^3/a^2)]$.

At $(\pm c, 0)$, $dy/dx = 2(c^2/a^2)$, numerically twice the slope at I.

$x = a^2/k$, $y = (a^4/k^3) - (c^2/k)$ are parametric equations for the graph.

Hence the following construction for the graph in Case (1):

Locate the inflection I; through it draw AIL and KIM parallel to the axes of abscissas and ordinates respectively, and in the positive direction in each case. Through I draw the inflectional tangent IQ, giving it the slope $-(c^2/a^2) = (1/3\alpha)(3\gamma - \beta^2)$. Make $AI = a$. From A draw AK to any point K of KM; thence $KL \perp AK$; thence $LM \perp KL$. On IL lay off $IN = KI$ (by rotating IK contraclockwise through $\pi/2$); through N draw $PQ \perp IL$. On MI lay off $MR = NQ$ (and in the same direction); through R draw $RP \perp IM$, thus determining a point P. [This construction is made in the figure for the series of points with subscript 7, and the points are indicated (without actually drawing the lines) for those having the subscripts 1, 2, 3 and 6.]



The point P is a point of the graph. The proof rests upon the theorem that if a perpendicular p is drawn from the vertex of a right triangle to the hypotenuse, dividing the hypotenuse into segments m and n, then $mn = p^2$. For if AK' (not shown in the figure) is drawn perpendicular to AK to meet IM at K', and

if IK' (not shown) $= k$, then $KI = a^2/k$; similarly $IL = KI^2/a = a^3/k^2$; $IM = IL^2/KI = a^4/k^3$. Also $IN = KI = a^2/k = x$; $MR = NQ = -(c^2/a^2) \times IN = -(c^2/k)$, and $IR = IM + MR = IM + NQ = (a^4/k^3) - (c^2/k) = y$.

Special Points. The minimum point (P_2 in the figure) can be determined geometrically, and there are two points, P_4 and P_5, for which the complete construction is not necessary.

If we make $CI = c$, and $\measuredangle ICK_2 = 30°$, the point obtained by the construction above will be the minimum point P_2, for then $x = IN = KI = c/\sqrt{3}$.

The intercept P_4 might be constructed by taking $K_4I = c$, but may be located (as in the figure) by laying off on IL, $IP_4 = c$.

If $k = a$, $KI = IL = IM = IN = a$. The corresponding point P_5 may be located by laying off IN_5 (which is also IL_5) $= a$, and Q_5P_5 through N_5, perpendicular to IL, and $= a$. [In any case, $y = IM + NQ = IM - QN$, so that P may always be located by measuring from Q a distance $y + QN = QP = IM$, parallel to the axis of ordinates.]

The Left Side.—The graph is symmetrical with regard to I, since $y \equiv f(x)$ is such that $f(-x_0) = -f(x_0)$; hence points determined on one side of I can be carried diametrically across to points on the other side. Points on the left side can also be determined directly by using negative values of k; that is, by locating K above I (K', not used in the construction, has the coördinates $(0, k)$).

Case (2). $\gamma - (\beta^2/3) > 0$. Write $\alpha = a^2$, $\gamma - (\beta^2/3) = c^2$; then the equation is

$$a^2y = x^3 + c^2x.$$

There is no extremum, and no intercept aside from the inflection I. $x = a^2/k$, $y = a^4/k^3 + c^2/k$ are parametric equations for the graph. If in the construction for Case (1), we make the slope of IQ c^2/a^2 instead of $-(c^2/a^2)$, the general description of the construction applies. [In either case, the slope of IQ is $[\gamma - (\beta^2/3)]$. The effect of the construction is to make the ordinate of P less than the ordinate of M in numerical value in Case (1), greater in numerical value in Case (2). The ordinate-difference, $QP = IM$, may be laid off directly from Q in either case.]

Case (3). $\gamma - (\beta^2/3) = 0$. The construction in Case (1) applies, though as in Case (2), there is no extremum, and no intercept except I. IQ coincides with IL, so that the ordinate of P is equal to the ordinate of M.

General Considerations. The Tangent at any Point.—The tangent at any point of the graph has for its y-intercept, $IT = -(2a^4/k^3) = -2IM$; for at any point $(a^2/k, a^4/k^3 \pm c^2/k)$ of $a^2y = x^3 \pm c^2x$, the slope is $3a^2/k^2 \pm c^2/a^2$, and the tangent itself has the equation

$$y = \left(\frac{3a^2}{k^2} \pm \frac{c^2}{a^2}\right)x - \frac{2a^4}{k^3}.$$

To draw the tangent at any point, P, that has been constructed, lay off $IT = -2IM$, join P and T; PT is the tangent.

The construction is shown for the tangents at P_1, P_3, P_4, P_5. T_2 coincides with R_2.

At diametrically opposite points, corresponding to equal and opposite values of k, the slope is the same; at two such points the tangents are parallel.

Graph for a Cubic Equation.—If the graph is to be used in connection with the solution of a cubic equation, the value of $\alpha = a^2$ is arbitrary, and may be so chosen as to separate or crowd the roots. The original axes may conveniently be located after the graph is drawn—with reference to the axes IL, IM, their intersection is at $(\beta/3, (1/27\alpha)(9\beta\gamma - 2\beta^3 - 27\delta))$. If it is convenient to make $a = c$, *i. e.*, $\alpha = \beta^2/3 - \gamma$, the construction will be somewhat simplified, as then $MR = IK$, and the point Q becomes superfluous.

The figure illustrates the construction for Case (1), showing seven points, with tangents, on each side of the inflection: the three special points P_2, P_4, P_5, a point in each of the segments IP_2 and P_2P_4, and two points beyond P_5.

In Case (2) or Case (3), fewer points would give an equally good guide for sketching the graph.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Dr. W. H. WILSON has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

Associate Professor W. B. FORD, of the University of Michigan, has been promoted to a full professorship of mathematics.

Miss PAULINE SPERRY, of Smith College, has accepted a position as instructor in mathematics at the University of California.

Assistant Professor L. A. RICE, of Syracuse University, has resigned to become instructor in mathematics at Tufts College.

Dr. IDA BARNEY, formerly instructor in Smith College, has been appointed professor of mathematics in Lake Erie College.

Professor W. A. GARRISON, of Union College, has been appointed professor of mathematics at King College, Bristol, Tennessee.

At the University of Nebraska, Miss L. RUNGE and Mr. A. BABBITT have been promoted to assistant professorships of mathematics.

Dr. A. W. HOBBS, of Johns Hopkins University, has been appointed instructor in mathematics at the University of North Carolina.

At the Southern Methodist University, Dallas, Texas, Associate Professor E. H. JONES has been promoted to a professorship of mathematics.

At the University of Toronto, Dr. SAMUEL BEATTY has been promoted to an assistant professorship of mathematics.